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Gauge Systems: Presymplectic and Group Action Formulations

M. C. Muñoz-Lecanda[1] and N. Román-Roy[1]

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The aim of this paper is to compare the two more standard geometrical formulations of gauge systems: the so-called *presymplectic formulation* and the formulation by *group actions*. We summarize the main features involved in them and prove that, at least locally, every presymplectic formulation can be interpreted in terms of group actions. The converse is also proved.

1. INTRODUCTION

Essentially, constrained systems appear in physics under two kinds of circumstances: (1) when the equations of motion are incompatible and/or undetermined, and (2) when the system has some kind of symmetry. There are two different descriptions within the framework of modern mathematical physics. One consists in using techniques of *presymplectic geometry*, and is used mainly in order to describe finite-dimensional dynamical systems. The other one uses the theory of *group actions* and is applied especially when infinite-dimensional systems are considered, although it is very interesting in many cases.

Next we describe briefly the main characteristics of both descriptions and discuss their possible equivalence.

3. PRESYMPLECTIC FORMULATION

The presymplectic formulation of constrained systems is a geometrization of the initial development of Dirac and Bergmann on singular systems

[1]Departamento de Matemática Aplicada y Telemática, Universidad Politécnica de Cataluña, Aptdo. 30002, E-08080 Barcelona, Spain.

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(Dirac, 1964). A deep explanation of the Hamiltonian and Lagrangian formalism can be found in Gotay *et al.* (1978), Gotay and Nester (1979, 1980), Cariñena *et al.* (1985), Bergvelt and de Kerf (1986), Muñoz-Lecanda and Román-Roy (1991), and references quoted therein.

In this formulation the initial data of the problem are a smooth differentiable manifold M_0 which constitutes the *initial phase space* of the system, a presymplectic form $\omega_0 \in \Omega^2(M_0)$, and a (closed) *canonical Hamiltonian 1-form* $\alpha_0 \in \Omega^1(M_0)$ or a *canonical Hamiltonian function* $h_0 \in C^\infty(M_0)$ (i.e., such that $\alpha_0 = dh_0$, at least locally). The equations of motion are written as

$$i(X_0)\omega_0 - dh_0 = 0, \qquad X_0 \in \mathscr{X}(M_0)$$

where $i(X_0)\omega_0$ means the inner product of the vector field X_0 and the form ω_0. Thus, the constrained system is specified by the triad (M_0, ω_0, h_0) which is called a *presymplectic dynamical system.*

These equations are in general incompatible except perhaps in a set of points of M_0, which, in the cases of interest, make a closed regular submanifold $j_C: C \hookrightarrow M_0$ for which a vector field X_0 tangent to C exists [and we denote this fact as $X_0 \in \underline{\mathscr{X}(C)}$ hereafter], such that the equation

$$[i(X_0)\omega_0 - dh_0]|_C = 0 \tag{2.1}$$

holds. C is the *final constraint submanifold* which inherits a presymplectic structure $\omega_C = j_C^*\omega_0$. Locally, this submanifold is defined by the vanishing of a set of functions $\{\zeta\} \subset C^\infty(M_0)$ called *constraints.*

The initial phase space M_0 can be a submanifold $j_0: M_0 \hookrightarrow \mathscr{M}$, where $(\mathscr{M}, \Omega)$ is a symplectic manifold.[2] Therefore the final constraint submanifold can also be considered as a submanifold of $(\mathscr{M}, \Omega)$, $j'_C: C \hookrightarrow \mathscr{M}$ and, for all $p \in C$, if $\dim \mathrm{T}_pC^\perp = k$ and $\dim \mathrm{T}_pC \cap \mathrm{T}_pC^\perp = l$, we say C is of class $(l, 2s = k - l)$ ($\mathrm{T}_pC^\perp$ denotes the *orthogonal symplectic complement* of T_pC in $\mathrm{T}_p\mathscr{M}$). Then, l and $2s$ are the maximal number of first- and second-class constraints, respectively, appearing in a local description of C. (Notice that if ζ is a first-class constraint, then its associated Hamiltonian vector field X_ζ is tangent to C; if ζ is second class, then X_ζ is not tangent to C.) In particular, if $s = 0$, we call C a *coisotropic submanifold* of $\mathscr{M}$. On the other hand, the symplectic manifold $(\mathscr{M}, \Omega)$ can also be constructed directly from C, using the presymplectic structure ω_C, in such a manner that C is coisotropically imbedded on it (*coisotropic imbedding theorem*) (Gotay,

[2]For instance, in the Hamiltonian formalism, many times $\mathscr{M}$ is the cotangent bundle T^*Q of the configuration space Q of the system.

1982; Marle, 1983). In any case, every symplectic manifold $(\mathcal{M}, \Omega)$ containing (C, ω_C) is called an *ambient symplectic manifold* or an *extended phase space* of (C, ω_C).

The solution of the system of equations (2.1) is not unique. The points of C reached from another fixed one $p \in C$ by means of integral curves of these solutions (passing through p) in the same lapse of the evolution parameter are the *gauge equivalent points* or *states* and the vector fields whose integral curves are made of gauge equivalent points are the *gauge fields*, which we denote $\mathcal{G}$. Under certain regularity conditions, it is proved that $\mathcal{G} = \underline{\ker \omega_C}$, where $\underline{\ker \omega_C}$ denotes the set of vector fields of $\mathcal{X}(M_0)$ whose restrictions to C make up $\ker \omega_C$ (that is, $j_{C*} \ker \omega_C = \underline{\ker \omega_C}|_C$).[3] In this case the dynamical problem posed by the equations $\overline{(2.1)}$ can be treated equivalently by studying the solution of the equation

$$j^*[i(X)\omega - dh] = 0 \tag{2.2}$$

with $X \in \mathcal{X}(M)$ tangent to C. [See Cariñena *et al.* (1985), Bergvelt and de Kerf (1986), Muñoz-Lecanda (1989), and Román-Roy (1988) for details.]

It is assumed that gauge equivalent states represent the same physical state. The geometrical procedure in order to eliminate this physical redundancy is to make the quotient of C by the foliation generated by the involutive distribution $\ker \omega_C$ [which is denoted $\mathcal{F}(\ker \omega_C)$]. Under suitable regularity conditions, the quotient space $\tilde{C}$ is a differentiable manifold, the projection $\pi\colon C \to \tilde{C}$ is a submersion, and $\tilde{C}$ is endowed with a symplectic structure $\tilde{\omega}$ such that $\pi^*\tilde{\omega} = \omega_C$. $(\tilde{C}, \tilde{\omega})$ is called the *manifold of physical states*, and equations (2.1) and (2.2) project in a natural way to $\tilde{C}$:

$$i(\tilde{X})\tilde{\omega} - d\tilde{h} = 0 \tag{2.3}$$

where $\pi^*\tilde{h} = j_C^* h_0$ and, for all $X_0 \in \underline{\mathcal{X}(C)}$ which is a solution of (2.1), $\pi_* X_0 = \tilde{X}$ is the unique solution of $\overline{(2.3)}$. Note that the existence of $\tilde{h}$ is assured because $i(\ker \omega_C)(j_C^* h_0) = 0$, as can be easily proved using the dynamical equation on C [equation (2.2)].

3. FORMULATION BY GROUP ACTIONS. TRANSLATION TO THE PRESYMPLECTIC FORMULATION

This formulation is based on the theory of actions of groups on symplectic manifolds (with well-defined momentum map) (Abraham and Marsden, 1978; Souriau, 1969; Warner, 1971; Libermann and Marle, 1987). A more detailed exposition of this description can be found in Marsden and Weinstein (1974), Weinstein (1979), and Gotay *et al.* (1990).

[3] $\underline{\ker \omega_C}$ is locally generated by vector fields which can be taken as Hamiltonian. In that case, their Hamiltonian functions are the *first-class constraints*.

The initial data of the formulation are now a smooth symplectic manifold $(\mathcal{M}, \Omega)$ which constitutes the extended phase space of the system, a Lagrangian function, or directly a Hamiltonian function $h \in C^\infty(\mathcal{M})$ [or a Hamiltonian 1-form $\alpha \in Z^1(M)$], and a group G of symmetries of the system $(\mathcal{M}, \Omega, h)$ called the *gauge group*, whose action in $(\mathcal{M}, \Omega)$ is *strongly symplectic* or *Hamiltonian* [hence, the Lie algebra $\mathbf{g}$ of G is realized in $\mathcal{X}(\mathcal{M})$ by means of Hamiltonian vector fields $\mathcal{X}_\mathbf{g}(\mathcal{M})$.

The next step is to construct a momentum map $J: \mathcal{M} \to \mathbf{g}^*$ associated to the action of G in $\mathcal{M}$, which is defined in the following way: for all $m \in \mathcal{M}$, $J(m) := \mu \in \mathbf{g}^*$ such that

$$J(m): \quad X \mapsto f_X(m)$$

where f_X is one Hamiltonian function associated to the vector field of $\mathcal{X}_\mathbf{g}(\mathcal{M})$ which realizes X.[4] We assume that J is *Ad-equivariant*, that is, the action of G is a *Poissonian action*. Then, let μ be a weakly regular value of the momentum map. The set $J^{-1}(\mu)$ is the *constraint submanifold*, $j_C: J^{-1}(\mu) \hookrightarrow \mathcal{M}$. In this way, the constrained system is entirely specified by $(\mathcal{M}, \Omega, h, G, J)$. Such a quintuple is usually called a *dynamical system with symmetry*.

Concerning the dynamics, if $X_h \in \mathcal{X}(\mathcal{M})$ is the Hamiltonian vector field associated to h, that is, a solution of the equation $i(X)\Omega - dh = 0$, then its trajectories with initial condition in $J^{-1}(\mu)$ are contained in this submanifold, since h is invariant by G, and thus $X_h \in \underline{\mathcal{X}(J^{-1}(\mu))}$. Hence, the component functions of J are invariant by X_h.

If we want to make a local description of $J^{-1}(\mu)$, we have that the constraints defining it locally are the component functions of $J = \mu$ (ctn). In fact, observe that, if $\{X_i\}$ is a base of $\mathbf{g}$, $\{\zeta_i\}$ are the Hamiltonian functions associated to these vector fields by the comomentum map, and $\{\alpha^i\}$ is the dual base in $\mathbf{g}^*$, then if $\mu = \mu_i \alpha^i$ with μ_i fixed constants, we have that

$$\begin{aligned} J^{-1}(\mu) &:= \{m \in \mathcal{M} \mid J(m) = \mu\} \\ &= \{m \in \mathcal{M} \mid J(m)(X) = \mu(X), \forall X \in \mathbf{g}\} \\ &= \{m \in \mathcal{M} \mid J(m)(X_i) = \mu_i\} \\ &= \{m \in \mathcal{M} \mid \zeta_i(m) = \mu_i\} \end{aligned}$$

and the constraints are $\zeta_i - \mu_i$.

[4]The map $J^*: X \mapsto f_X$ is called the *comomentum map*.

At this point, one can observe that, taking $M_0 = J^{-1}(\mu)$, the natural imbedding $j_0: M_0 \hookrightarrow \mathcal{M}$, the form $\omega_0 = j_0^* \Omega$, and the function $h_0 = j_0^* h$, then $(J^{-1}(\mu), \omega_0, h_0)$ is a presymplectic system. This system is compatible since the equations

$$i(X_0)\omega_0 - dh_0 = 0 \tag{3.1}$$

have the vector field $X_0 \in \mathscr{X}(M_0)$ such that $j_{0_*} X_0 = X_h|_{M_0}$ as a solution [remember that X_h is tangent to $J^{-1}(\mu)$]. Hence, M_0 is, in this case, the final constraint submanifold C of this presymplectic system.

In this formulation, gauge redundancy can be discussed in the following terms: if h is a Hamiltonian function invariant by G, so is $h' := h + \zeta$, for all constraints ζ defining locally $M_0 = J^{-1}(\mu)$, and both restricted to h_0 in M_0. This means that X_h and $X_{h'}$ are valid solutions for the dynamics and, as one can observe, the corresponding vector fields X_0 and X_0' in $\mathscr{X}(M_0)$, which are solutions of (3.1), differ in an element of $\ker \omega_0$.

Now, let G_μ be the isotropy group of $J^{-1}(\mu)$. By equivariance, $J^{-1}(\mu)$ is stable under the action of G_μ. Therefore the quotient $J^{-1}(\mu)/G_\mu$ is well defined and is called the *orbit space* of $J^{-1}(\mu)$. Then, if the action of G_μ in $J^{-1}(\mu)$ is proper and free, the orbit space is a differentiable manifold which is endowed with a (unique) symplectic structure $\tilde{\omega}$ (*Marsden–Weinstein theorem*). This is the manifold of physical states in this formulation. It is clear that h projects onto $J^{-1}(\mu)/G_\mu$, since h is invariant under the action of G, and the same thing holds for the dynamical equation.

Observe that the set of vector fields of $\mathbf{g}$ which are tangent to G_μ (i.e., the Lie algebra of G_μ), which we denote $\mathbf{g}_\mu \subset \mathbf{g}$, is realized in $\mathscr{X}(\mathcal{M})$ by means of vector fields which are the gauge vector fields $\mathscr{G}$ of the first formulation and, hence, the manifold of physical states are the same in both formulations. In fact, since G_μ is the isotropy group and letting $J^{-1}(\mu)$ be invariant, $\mathbf{g}_\mu$ is realized by Hamiltonian vector fields which are tangent to $J^{-1}(\mu)$ (denote this set by $\mathscr{G}$). Therefore, their Hamiltonian functions are first-class constraints necessarily and, hence $\mathscr{G} \subseteq \underline{\ker \omega_C}$. But since the final constraint submanifolds C and $J^{-1}(\mu)$ are identified and the quotient spaces $C/\mathscr{F}(\ker \omega_C)$ and $J^{-1}(\mu)/G_\mu$ have to be symplectic manifolds, this implies that $\mathscr{G} = \underline{\ker \omega_C}$ (i.e., $\mathscr{G}$ is the set of gauge vector fields) and the quotients are equal.

Summarizing, we have proved the following result:

Theorem 1. Every system with symmetry $(\mathcal{M}, \Omega, h, G, J)$ can be described as a presymplectic dynamical system (M_0, ω_0, h) , *where the final constraint submanifold is* $J^{-1}(\mu) = M_0$ and the Lie algebra $\mathbf{g}_\mu$ of the isotropy group of $J^{-1}(\mu)$ is realized by the gauge group $\mathscr{G} = \ker \omega_0$.

4. THE PRESYMPLECTIC FORMULATION AS A GROUP ACTION FORMULATION

In the last section, we concluded that the description of a dynamical system with symmetry can be interpreted as a presymplectic dynamical system where the isotropy group of $J^{-1}(\mu)$ is identified with the gauge group. Now, we pose the converse question; it is possible to interpret any presymplectic system as a system with symmetry? In other words, can we formulate the presymplectic formulation of a constrained system in terms of group actions? The answer is affirmative, at least locally, and the result we obtain is the following:

Theorem 2. Let (M_0, ω_0, h_0) be a presymplectic dynamical system with final constraint submanifold C, and let $(\mathcal{M}, \Omega)$ be an ambient symplectic manifold such that $j: C \hookrightarrow \mathcal{M}$ is a submanifold of class $(l, 2s)$ in $\mathcal{M}$. Let $X \in \mathcal{X}(\mathcal{M})$ be a vector field tangent to C, solution of equation (2.2).

If $p \in C$, then there exists an open set U in $\mathcal{M}$ such that $p \in U$, and an Abelian Lie group G such that:

1. G acts freely on the points of U.
2. The action of G is strongly symplectic and there exists a momentum map J associated to it.
3. $C \cap U = J^{-1}(0)$.
4. The Lie algebra $\mathbf{g}_0$ of the isotropy group G_0 of C is realized in the gauge group $\ker \omega_C$.

Proof. Let $\dim \mathcal{M} = 2n$ and $p \in C$. According to Shanmugadhasan (1973), there exists an open set U of p in $\mathcal{M}$ and a local system of coordinates $\varphi = \{q^i, p_i, \eta^j, \xi_j, \chi^k, \chi'_k\}$ (with $k = 1, \ldots, s$; $j = 1, \ldots, l$; $i = 1, \ldots, n - s - l$), such that

1. $\varphi(U) = \mathbf{R}^{2n}$, $\varphi(p) = 0$.
2. $\Omega|_U = dq^i \wedge dp_i + d\eta^j \wedge d\xi_j + d\chi^k \wedge d\chi'_k$.
3. $C \cap U = \{x \in U | \xi_i(x) = 0,\ \chi^k(x) = 0,\ \chi'_k(x) = 0\}$.

Then $j^*\Omega|_{C \cap U} = dq^i \wedge dp_i$, and $[\ker(j^*\Omega)]|_{C \cap U}$ is spanned by $\{\partial/\partial\eta^j\}$. Observe that the functions ξ_i are first-class constraints and χ^k, χ'_k are second-class constraints.

Let $\mathcal{D}$ be the distribution in U spanned by $\{\partial/\partial\eta^j, \partial/\partial\chi^k, \partial/\partial\chi'_k\}$.[5] $\mathcal{D}$ is an involutive distribution and the Lie brackets of elements of this local base are zero. The submanifolds defined by

$$\{x \in U | q^i(x) = \text{ctn},\ p_i(x) = \text{ctn},\ \xi_j(x) = \text{ctn}\}$$

[5] Observe that these vector fields are a local base of $\mathcal{X}(C)^{\perp}$, the set of vector fields taking values in $TC^{\perp}$.

are maximal integral manifolds of $\mathscr{D}$. All of them are diffeomorphic and have the structure of an Abelian Lie group defined in the following way: consider $U_0 = \{x \in U | q^i(x) = a^i,\ p_i(x) = a_i,\ \xi_j(x) = b_j\}$ (with $a^i, a_i, b_j \in \mathbf{R}$). The map

$$\psi:\quad U_0 \to \mathbf{R}^{l+2s}$$
$$x \mapsto (\eta^j(x), \chi^k(x), \chi'_k(x))$$

is a diffeomorphism which translates the structure of the Lie group from $\mathbf{R}^{l+2s}$ to U_0. Then each one of these maximal integral manifolds of $\mathscr{D}$ have a natural structure of an Abelian Lie group.

Consider $G = \mathbf{R}^{l+2s}$ with the usual Lie group structure. Then, G acts freely in U in the natural way: if $x \in U$ and $g \in G$, then $gx = \varphi^{-1}(\varphi(x) + g)$.

This action has the following properties:

1. It is strongly symplectic because

$$L\left(\frac{\partial}{\partial \eta^j}\right)\Omega = L\left(\frac{\partial}{\partial \chi^k}\right)\Omega = L\left(\frac{\partial}{\partial \chi'_k}\right)\Omega = 0$$

and the vector field $\partial/\partial\eta^j$, $\partial/\partial\chi^k$, $\partial/\partial\chi'_k$ are Hamiltonian.

2. It preserves the Hamiltonian function h on $C \cap U$. In fact, if $x \in C \cap U$, then, for all j,

$$\frac{\partial h}{\partial \eta^j}(x) = \Omega\left(\frac{\partial}{\partial \eta^j}, X_h\right)\bigg|_x = 0$$

because $\partial/\partial\eta^j \in \mathscr{X}(C)^{\perp}$ and X_h is tangent to C. Now we have to prove it for $\partial/\partial\chi^k$ and $\partial/\partial\chi'_k$ (for all k), but it holds since

$$0 = X_h(\chi^k)|_C = -\frac{\partial h}{\partial \chi'_k}\bigg|_C$$

and the same thing for $\partial/\partial\chi^k$.

3. There exists a momentum map. In fact, a *comomentum map* J^* can be defined as

$$J^*\left(\frac{\partial}{\partial \eta^j}\right) = \xi_j, \qquad J^*\left(\frac{\partial}{\partial \chi^k}\right) = \chi'_k, \qquad J^*\left(\frac{\partial}{\partial \chi'_k}\right) = \chi^k$$

Then the momentum map is given by $J := (\xi_j, \chi'_k, \chi^k)$ and therefore $C \cap U = J^{-1}(0)$.

4. The Lie algebra $\mathbf{g}_0$ of the isotropy group G_0 of C is realized by the elements of $\mathscr{D}$ whose integral curves are contained in C, which are $\{\partial/\partial\eta^j\}$, that is, $\ker(j^*\Omega) = \ker \omega_C$. ■

Observe that if (M, ω) is a symplectic manifold in which (C, ω_C) is coisotropically imbedded, $C \hookrightarrow M \hookrightarrow \mathscr{M}$, then locally the situation is the following: the group G' acting on M is G_0, the isotropy group of C.

5. COMMENTS AND DISCUSSION

When we compare both formulations, some remarks can be pointed out:

1. In the first case, the final constraint submanifold is not a datum of the problem, but it is obtained from the analysis of the compatibility of the equations of motion and the stability of their solutions. In the second one, the final constraint submanifold arises immediately as a level set of the momentum map and dynamics is not required in order to achieve it.

2. In the second formulation, additional information is needed in relation to the first one: the gauge group G has to be known in the beginning, and, from it, the isotropy group G_μ can be obtained. In the first formulation, G_μ would be obtained once the constraint algorithm is finished and the final constraint submanifold is known: it would be the Lie group which has $\mathcal{G} \cong \mathbf{g}_\mu$ as Lie algebra.

3. Notice that, in the particular case $G_\mu = G$, we have $\mathbf{g}_\mu = \mathbf{g}$ and, according to the last comment, this means that all the constraints appearing in the theory are first class. If $G_\mu \subset G$, then it means that there are also second-class constraints. This last situation is physically undesirable, since these constraints represent physically irrelevant degrees of freedom, which can be obviated and must be suppressed in order to obtain a correct interpretation of the dynamics and a possible quantization of the system.

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